

# Quantum deformation of conformal symmetry in $N = 4$ super Yang-Mills theory

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## Abstract

In gauge theories, not all rigid symmetries of the classical action can be maintained manifestly in the quantization procedure, even in the absence of anomalies. If this occurs for an anomaly-free symmetry, the effective action is invariant under a transformation that differs from its classical counterpart by quantum corrections. As shown by Fradkin and Palchik years ago, such a phenomenon occurs for conformal symmetry in quantum Yang-Mills theories with vanishing beta function, such as the  $\mathcal{N} = 4$  super Yang-Mills theory. More recently, Jevicki et al demonstrated that the quantum deformation of conformal symmetry sheds light on the nature of the AdS/CFT correspondence. In this paper, we derive the conformal Ward identity for the bosonic sector of the  $\mathcal{N} = 4$  super Yang-Mills theory using the background field method. We then compute the leading quantum deformation of the conformal transformation for a specific Abelian background which is of interest in the context of the AdS/CFT correspondence. In the case of scalar fields, our final result agrees with that of Jevicki et al. The resulting vector and scalar transformations coincide with those which are characteristic of a D3-brane embedded in  $AdS_5 \times S^5$ .

# 1 Introduction

$\mathcal{N} = 4$  super Yang-Mills theory was popular in the 1980's as the unique maximally supersymmetric Yang-Mills theory in four space-time dimensions, and the first ultraviolet-finite quantum field theory ever constructed. More recently, it has become the subject of immense scrutiny in the context of the AdS-CFT correspondence [1, 2, 3, 4]. Yet there is another interesting field theoretic aspect of this dynamical system which has so far not received much attention, except in [5], and which sheds light on the origin of the AdS-CFT correspondence. Quantum conformal invariance in the  $\mathcal{N} = 4$  super Yang-Mills theory turns out to be a nontrivial deformation of the linear conformal symmetry of the classical action, and this fact has interesting implications.

In 1984, it was shown by Fradkin and Palchik [6] (see also [7, 8]) that the generating functional in conformally invariant quantum non-Abelian gauge theories is not invariant under standard (linearly realized) special conformal transformations; such an invariance is only consistent with a purely longitudinal two-point function,  $\langle A_m(x_1) A_n(x_2) \rangle \propto \partial_m \partial_n \ln(x_1 - x_2)^2$ . These theories are, however, invariant under *deformed* special conformal transformations consisting of a combination of conformal transformations and compensating field-dependent gauge transformations; the conformal Ward identity associated with the deformed symmetry leads to the correct transverse propagator.

In 1998, Jevicki *et al* [5] applied and extended the Fradkin-Palchik construction to address a question that may be formulated as follows: Where are the branes implied by the AdS/CFT duality within the framework of super Yang-Mills theory? Using a derivative expansion of the effective action, they computed the leading quantum deformation of the conformal transformation law of the Higgs fields  $Y_\mu$ , with  $\mu = 1, \dots, 6$ , which trigger the spontaneous breakdown of the gauge group  $SU(N+1)$  to  $SU(N) \times U(1)$  in the  $\mathcal{N} = 4$  super Yang-Mills theory. The deformed transformation is

$$\delta Y_\mu = -\delta x^m \partial_m Y_\mu + 2(b \cdot x) Y_\mu, \quad \delta x^m = b^m x^2 - 2x^m (b \cdot x) Y_\mu + b^m \frac{R^4}{Y^2}, \quad (1.1)$$

where  $Y^2 = Y_\mu Y_\mu$  and  $R^4 = N g_{\text{YM}}^2 / (2\pi^2)$ , and coincides with that of the transverse degrees of freedom of a D3-brane embedded in  $AdS_5 \times S^5$ . In conjunction with the requirement of  $SO(6)$  invariance, the above transformation uniquely fixes the part of the low energy effective action for the D3-brane which depends on the scalar fields and their first derivatives only [1],

$$S = -\frac{1}{g_{\text{YM}}^2 R^4} \int d^4x Y^4 \left( \sqrt{-\det(\eta_{mn} + R^4 \partial_m Y \cdot \partial_n Y / Y^4)} - 1 \right). \quad (1.2)$$

The coupling constant  $g_{\text{YM}}^2$  can be treated as a loop-counting parameter in  $\mathcal{N} = 4$  super Yang-Mills theory. In this respect, the  $R^4$  dependent term in the transformation  $\delta Y$  in eq. (1.1) is just a one-loop quantum deformation. On the other hand, the action (1.2) is the result of summing up quantum corrections to all loop orders. Therefore, even the one-loop symmetry deformation contains essential information about the structure of the effective action at higher loops! This is one of the main reasons why we consider it important to pursue the study of quantum deformations of (super) conformal symmetry in finite  $\mathcal{N} = 2, 4$  super Yang-Mills theories.

The story with conformal symmetry is an example of the more general phenomenon of quantum deformations of rigid symmetries in gauge theories, which in fact embraces two different aspects: (i) deformations before gauge fixing; and (ii) deformations after gauge fixing. Point (i) concerns the problem of extending any rigid symmetry of a classical gauge invariant action to the ghosts and the antifields, in the framework of the Batalin-Vilkovisky quantization scheme [9] and its extensions, so as to leave the solution of the master equation invariant. It is always possible to construct a solution to this problem [10] in which the global symmetry is (antibracket) canonically realized. Point (ii) has been analysed in a rather general setting by van Holten [11], and here we simply quote his formulation of the problem: it does not hold that “any rigid symmetry of a classical action can always be maintained manifestly in the Faddeev-Popov–BRST quantization scheme, even in the absence of anomalies.” Examples include on-shell supersymmetry [12], conformal symmetry [6] and its supersymmetric extensions. According to the analysis of [11], the problem of keeping the rigid symmetries manifest at the quantum level is essentially equivalent to finding covariant gauge conditions. In the case of conformal symmetry, such gauge conditions do not exist [6] and any special conformal transformation has to be accompanied by a field-dependent nonlocal gauge transformation in order to restore the gauge slice.

The present paper is organized as follows. In section 2, we give a general discussion of quantum deformations of global symmetries (for a general gauge theory) in the framework of DeWitt–Faddeev-Popov–BRST quantization. This section is of a review nature and consists largely of variations on themes suggested by Fradkin and Palchik, van Holten and others. In section 3, we derive the conformal Ward identity in the  $\mathcal{N} = 4$  super Yang-Mills theory within the background field method. The analysis in this paper is restricted to the (full) bosonic sector of the theory, simply because the fermions are not relevant for the subsequent considerations; they can be included by simple extension. The content of section 3 significantly extends the earlier results for scalar fields given in

[5]. In addition, our derivation of the conformal Ward identity is more rigorous. It is worth mentioning that the content of section 3 can easily be generalized to the case of finite  $\mathcal{N} = 2$  super Yang-Mills models. In section 4, we compute the leading quantum deformation of the conformal transformation for a specific Abelian background which is of interest in the context of the AdS/CFT correspondence. The resulting vector and scalar transformations coincide with those characteristic of a D3-brane embedded in  $AdS_5 \times S^5$ . The results and interesting open problems are then discussed in section 5.

## 2 Symmetries of the effective action

In this section, we use DeWitt's condensed notation [13] (see also [14]), and for simplicity restrict attention to the case of bosonic gauge theories. Let  $S[\Phi]$  be the action of an irreducible gauge theory (following the terminology of [9]) involving bosonic fields  $\Phi^i$ . By definition, the gauge generators  $R^i_\alpha[\Phi]$  give rise to Noether identities

$$S_{,i}[\Phi] R^i_\alpha[\Phi] \equiv 0 , \quad (2.1)$$

and in what follows they are assumed to form a closed algebra,

$$R^i_{\alpha,j}[\Phi] R^j_\beta[\Phi] - R^i_{\beta,j}[\Phi] R^j_\alpha[\Phi] = R^i_\gamma[\Phi] f^\gamma_{\alpha\beta}[\Phi] , \quad (2.2)$$

together with additional requirements

$$R^i_{\alpha,i}[\Phi] = 0 , \quad f^\beta_{\alpha\beta}[\Phi] = 0 . \quad (2.3)$$

It will be also assumed that the gauge transformations

$$\delta\Phi^i = R^i_\alpha[\Phi] \delta\zeta^\alpha , \quad (2.4)$$

with  $\delta\zeta^\alpha$  arbitrary parameters of compact support, span the gauge freedom of the theory – that is, if  $\Phi_0$  is a stationary point of the action,  $S_{,i}[\Phi_0] = 0$ , then the equality  $R^i_\alpha[\Phi_0] \delta\zeta^\alpha = 0$  implies  $\delta\zeta^\alpha = 0$ .

Under the above assumptions, the in-out vacuum amplitude is known to have a functional integral representation of the form

$$\langle \text{out} | \text{in} \rangle = N \int d\Phi \text{Det}(F[\Phi]) e^{i(S[\Phi] + S_{\text{GF}}[\chi[\Phi]])} , \quad (2.5)$$

where  $\chi^\alpha[\Phi]$  are gauge conditions such that the operator

$$F^\alpha_\beta[\Phi] \equiv \chi^\alpha_{,i}[\Phi] R^i_\beta[\Phi] \quad (2.6)$$

is non-singular at  $\Phi_0$ . The gauge fixing functional  $S_{\text{GF}}[\chi]$  is chosen in such a way that the action  $S[\Phi] + S_{\text{GF}}[\chi[\Phi]]$  is no longer gauge invariant. In perturbation theory, it is customary to choose  $S_{\text{GF}}[\chi]$  to be of Gaussian form,  $S_{\text{GF}}[\chi] = \frac{1}{2}\chi^\alpha \eta_{\alpha\beta} \chi^\beta$ , with  $\eta_{\alpha\beta}$  a constant non-singular symmetric matrix.

The in-out vacuum amplitude (2.5) is independent of the choice of  $\chi$ ,  $\langle \text{out} | \text{in} \rangle_{\chi+\delta\chi} = \langle \text{out} | \text{in} \rangle_\chi$ , with  $\delta\chi^\alpha[\Phi]$  a small deformation of the gauge conditions. It is important for subsequent considerations to recall an old proof<sup>1</sup> of this fact due to DeWitt [13, 15]. In the functional integral

$$\langle \text{out} | \text{in} \rangle_{\chi+\delta\chi} = N \int d\tilde{\Phi} \text{Det}(F[\tilde{\Phi}] + \delta F[\tilde{\Phi}]) e^{i(S[\tilde{\Phi}] + S_{\text{GF}}[\chi[\tilde{\Phi}] + \delta\chi[\tilde{\Phi}]])} , \quad (2.7)$$

with  $\delta F^\alpha_\beta[\Phi] = \delta\chi^\alpha_{,i}[\Phi] R^i_\beta[\Phi]$ , we make a replacement of integration variables

$$\tilde{\Phi}^i = \Phi^i - R^i_\alpha[\Phi] \delta\zeta^\alpha[\Phi], \quad \delta\zeta^\alpha[\Phi] = (F^{-1}[\Phi])^\alpha_\beta \delta\chi^\beta[\Phi] \quad (2.8)$$

chosen so that

$$S[\tilde{\Phi}] + S_{\text{GF}}[\chi[\tilde{\Phi}] + \delta\chi[\tilde{\Phi}]] = S[\Phi] + S_{\text{GF}}[\chi[\Phi]].$$

On the other hand, direct calculations yield

$$\begin{aligned} d\tilde{\Phi} \text{Det}(F[\tilde{\Phi}] + \delta F[\tilde{\Phi}]) &= d\Phi \text{Det}(F[\Phi]) \left\{ 1 - R^i_{\alpha,i}[\Phi] \delta\zeta^\alpha[\Phi] - f^\beta_{\beta\alpha}[\Phi] \delta\zeta^\alpha[\Phi] \right\} \\ &= d\Phi \text{Det}(F[\Phi]) , \end{aligned} \quad (2.9)$$

as a consequence of (2.3). Similar considerations can be used to show that the correlation function  $\langle \text{out} | \Psi[\Phi] | \text{in} \rangle$  of a gauge invariant functional  $\Psi[\Phi]$ , with  $\Psi_{,i}[\Phi] R^i_\alpha[\Phi] = 0$ , is not dependent on the gauge choice.

Next, we turn our attention to the effective action of the theory,

$$\Gamma[\phi] = (W[J] - J_i \phi^i)|_{J=J[\phi]} , \quad \phi^i = \frac{\delta}{\delta J_i} W[J] , \quad (2.10)$$

with  $W[J]$  the generating functional of connected Green's functions,

$$e^{iW[J]} = N \int d\Phi \text{Det}(F[\Phi]) e^{i(S[\Phi] + S_{\text{GF}}[\chi[\Phi]] + J_i \Phi^i)} . \quad (2.11)$$

The effective action depends explicitly on the choice of the gauge condition  $\chi[\Phi]$ , unlike the  $S$ -matrix following from  $\Gamma[\phi]$ . It is not, however, the issue of gauge dependence which

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<sup>1</sup>Similar arguments can be used to establish the independence of the vacuum to vacuum amplitude on the functional form of  $S_{\text{GF}}$ ; see [15]. The inclusion of Nielsen-Kallosh ghosts [16] may be important.

is the point of concern here. Suppose the classical action is invariant,  $S[\Phi + \epsilon \Omega[\Phi]] = S[\Phi]$ , under a rigid transformation

$$\delta \Phi^i = \epsilon \Omega^i[\Phi] , \quad (2.12)$$

with  $\epsilon$  an infinitesimal constant parameter. We will analyse the implications of this classical symmetry for the effective action; see Ref. [11] for a similar earlier treatment. In what follows, some additional properties of the structure of the gauge and global transformations will be assumed, namely

$$\Omega^i_{,i}[\Phi] = 0 , \quad (2.13)$$

$$R^i_{\alpha,j}[\Phi] \Omega^j[\Phi] - \Omega^i_{,j}[\Phi] R^j_{\alpha}[\Phi] = R^i_{\beta}[\Phi] f^{\beta}_{\alpha}[\Phi] , \quad (2.14)$$

$$f^{\alpha}_{\alpha}[\Phi] = 0 . \quad (2.15)$$

Eq. (2.13) ensures that the Jacobian of the transformation  $\Phi^i \rightarrow \Phi^i + \epsilon \Omega^i[\Phi]$  is equal to one. Eq. (2.14) implies that the commutator of a gauge transformation with a global symmetry transformation is a gauge transformation.

To understand the manifestations of the symmetry (2.12) at the quantum level, we make the change of variables

$$\Phi^i = \tilde{\Phi}^i + \epsilon \Omega^i[\tilde{\Phi}] \quad (2.16)$$

in the right hand side of (2.11). Using eqs. (2.14) and (2.15), one then obtains

$$F^{\alpha}_{\beta}[\Phi] = F^{\alpha}_{\beta}[\tilde{\Phi}] + \delta_{\epsilon} F^{\alpha}_{\beta}[\tilde{\Phi}] + \epsilon F^{\alpha}_{\gamma}[\tilde{\Phi}] f^{\gamma}_{\beta}[\tilde{\Phi}] , \quad (2.17)$$

$$\text{Det}(F[\Phi]) = \text{Det}(F[\tilde{\Phi}] + \delta_{\epsilon} F[\tilde{\Phi}]) , \quad (2.18)$$

where

$$\delta_{\epsilon} F^{\alpha}_{\beta}[\tilde{\Phi}] = \delta_{\epsilon} \chi^{\alpha}_{,i}[\tilde{\Phi}] R^i_{\beta}[\tilde{\Phi}] , \quad \delta_{\epsilon} \chi^{\alpha}[\tilde{\Phi}] = \epsilon \chi^{\alpha}_{,i}[\tilde{\Phi}] \Omega^i[\tilde{\Phi}] .$$

As a result, eq. (2.11) becomes

$$\begin{aligned} e^{iW[J]} &= N \int d\tilde{\Phi} \text{Det}(F[\tilde{\Phi}] + \delta_{\epsilon} F[\tilde{\Phi}]) \\ &\times \exp i \left\{ S[\tilde{\Phi}] + S_{\text{GF}}[\chi[\tilde{\Phi}] + \delta_{\epsilon} \chi[\tilde{\Phi}]] + J_i(\tilde{\Phi}^i + \epsilon \Omega^i[\tilde{\Phi}]) \right\} . \end{aligned} \quad (2.19)$$

In the functional integral obtained, we can then change variables according to the rule (2.8) with  $\delta \chi[\Phi] = \delta_{\epsilon} \chi[\Phi]$ . This leads to the Ward identity

$$\Gamma_{,i}[\phi] \langle \Omega^i[\Phi] \rangle = \Gamma_{,i}[\phi] \langle R^i_{\alpha}[\Phi] (F^{-1}[\Phi])^{\alpha}_{\beta} \chi^{\beta}_{,j}[\Phi] \Omega^j[\Phi] \rangle , \quad (2.20)$$

where we have used the fact that  $J_i = -\Gamma_{,i}[\phi]$ . In the case of non-gauge theories, the right hand side of eq. (2.20) vanishes; see, for example, Weinberg's book [17]. In eq. (2.20), the symbol  $\langle \ \rangle$  denotes the quantum average in the presence of the source  $J = J[\phi]$ ,

$$\langle A[\Phi] \rangle = e^{-iW[J]} N \int d\Phi A[\Phi] \text{Det}(F[\Phi]) e^{i(S[\Phi] + S_{\text{GF}}[\chi[\Phi]] + J_i \Phi^i)} . \quad (2.21)$$

For a large class of gauge theories, the above Ward identity can be brought into a simpler form. Suppose that the gauge fixing functional is invariant,  $S_{\text{GF}}[\chi + \delta_\Lambda \chi] = S_{\text{GF}}[\chi]$ , under a linear homogeneous transformation

$$\delta_\Lambda \chi^\alpha = \epsilon \Lambda^\alpha{}_\beta \chi^\beta , \quad (2.22)$$

with  $\Lambda^\alpha{}_\beta$  a field independent operator. Since  $\Lambda$  is field independent, we have

$$e^{iW[J]} = N \int d\tilde{\Phi} \text{Det}(F[\tilde{\Phi}] + \delta_\Lambda F[\tilde{\Phi}]) e^{i(S[\tilde{\Phi}] + S_{\text{GF}}[\chi[\tilde{\Phi}] + \delta_\Lambda \chi[\tilde{\Phi}]] + J_i \tilde{\Phi}^i)} ,$$

where  $\delta_\Lambda F^\alpha{}_\beta[\tilde{\Phi}] = \delta_\Lambda \chi^\alpha{}_{,i}[\tilde{\Phi}] R^i{}_\beta[\tilde{\Phi}]$ . Replacing the integration variables according to the rule (2.8) with  $\delta\chi[\Phi] = \delta_\Lambda \chi[\Phi]$ , one then obtains

$$\Gamma_{,i}[\phi] \langle R^i{}_\alpha[\Phi] (F^{-1}[\Phi])^\alpha{}_\beta \Lambda^\beta{}_\gamma \chi^\gamma[\Phi] \rangle = 0 . \quad (2.23)$$

Our last assumption concerns the behaviour of the gauge conditions under the symmetry transformation. We assume

$$\delta_\epsilon \chi^\alpha[\Phi] \equiv \epsilon \chi^\alpha{}_{,i}[\Phi] \Omega^i[\Phi] = \epsilon \left( \Lambda^\alpha{}_\beta \chi^\beta[\Phi] + \rho^\alpha[\Phi] \right) , \quad \rho^\alpha[\Phi] \neq 0 , \quad (2.24)$$

where the homogeneous term on the right hand side leaves  $S_{\text{GF}}[\chi]$  invariant,  $S_{\text{GF}}[\chi^\alpha + \epsilon \Lambda^\alpha{}_\beta \chi^\beta] = S_{\text{GF}}[\chi^\alpha]$ . In this case, the Ward identity (2.20) is equivalent to

$$\Gamma_{,i}[\phi] \langle \Omega^i[\Phi] \rangle = \Gamma_{,i}[\phi] \langle R^i{}_\alpha[\Phi] (F^{-1}[\Phi])^\alpha{}_\beta \rho^\beta[\Phi] \rangle . \quad (2.25)$$

It is worth discussing the transformation law (2.24). The gauge conditions  $\chi^\alpha[\Phi] = 0$  break the gauge invariance and single out a unique representative from each gauge orbit. For most global symmetries, there exist *covariant* gauge conditions - that is, they can be chosen in such a way that  $\chi^\alpha[\Phi]$  transforms as in eq. (2.24) but with  $\rho^\alpha[\Phi] = 0$ . In this case, the global symmetry leaves the gauge slice  $\chi^\alpha[\Phi] = 0$  invariant. However, for some symmetries, such as conformal invariance, there is no way to eliminate the inhomogeneous term in (2.24), and the symmetry transformation does not leave the gauge conditions  $\chi^\alpha[\Phi] = 0$  invariant. In such a situation, a non-trivial symmetry deformation occurs at

the quantum level, as follows from eq. (2.25). Indeed, consider the simplest situation of a linearly realized classical symmetry,  $\Omega^i[\Phi] = \Omega^i_j \Phi^j$ . In this case, the left hand side in (2.25) is simply  $\Gamma_{,i}[\phi] \Omega^i[\phi]$ , and hence eq. (2.25) can be interpreted as the invariance condition under deformed transformations of the form  $\delta\phi^i = \epsilon \Omega^i[\phi] + \text{quantum corrections}$ .

It is instructive to re-derive the above results in the BRST approach [18], in which eq. (2.11) is replaced by

$$e^{iW[J]} = N \int d\Phi d\bar{C} dC e^{i(S_{\text{eff}}[\Phi, \bar{C}, C; \chi[\Phi]] + J_i \Phi^i)}, \quad (2.26)$$

where

$$S_{\text{eff}}[\Phi, \bar{C}, C; \chi[\Phi]] = S[\Phi] + S_{\text{GF}}[\chi[\Phi]] + \bar{C}_\alpha \chi^\alpha_{,i}[\Phi] R^i_\beta[\Phi] C^\beta, \quad (2.27)$$

with  $\bar{C}_\alpha$  and  $C^\alpha$  the Faddeev-Popov ghosts. The action  $S_{\text{eff}}$  is invariant under the following BRST transformation

$$\delta\Phi^i = R^i_\alpha[\Phi] C^\alpha \lambda, \quad \delta C^\alpha = \frac{1}{2} f^\alpha_{\beta\gamma}[\Phi] C^\gamma C^\beta \lambda, \quad \delta\bar{C}_\alpha = S_{\text{GF},\alpha}[\chi[\Phi]] \lambda, \quad (2.28)$$

with  $\lambda$  a constant anticommuting parameter. This BRST transformation leaves the integration measure  $d\Phi d\bar{C} dC$  in (2.26) invariant as a consequence of (2.3).

Inspired by [19], we make a BRST-like change of variables with a field dependent parameter in the right hand side of (2.26),

$$\begin{aligned} \Phi^i &\rightarrow \Phi^i + R^i_\alpha[\Phi] C^\alpha \lambda, & C^\alpha &\rightarrow C^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma}[\Phi] C^\gamma C^\beta \lambda, & \bar{C}_\alpha &\rightarrow \bar{C}_\alpha + S_{\text{GF},\alpha}[\chi[\Phi]] \lambda, \\ & & \lambda &= -\bar{C}_\alpha \delta\chi^\alpha[\Phi], \end{aligned} \quad (2.29)$$

where  $\delta\chi^\alpha[\Phi]$  are arbitrary variations of the gauge conditions. This transformation obviously leaves  $S_{\text{eff}}$  invariant, but the corresponding Jacobian is now non-trivial, and eq. (2.26) turns into

$$\begin{aligned} e^{iW[J]} &= N \int d\Phi d\bar{C} dC e^{i(S_{\text{eff}}[\Phi, \bar{C}, C; \chi[\Phi] + \delta\chi[\Phi]] + J_i \Phi^i)} \\ &\times \left( 1 - i J_i R^i_\alpha[\Phi] C^\alpha \bar{C}_\beta \delta\chi^\beta[\Phi] \right). \end{aligned} \quad (2.30)$$

For  $J = 0$ , this relation is equivalent to the gauge-independence of the in-out vacuum amplitude.

Given a rigid symmetry defined by eqs. (2.12) – (2.15), one can consider the following change of variables

$$\Phi^i \rightarrow \Phi^i + \epsilon \Omega^i[\Phi], \quad C^\alpha \rightarrow C^\alpha - \epsilon f^\alpha_{\beta}[\Phi] C^\beta, \quad \bar{C}_\alpha \rightarrow \bar{C}_\alpha \quad (2.31)$$



in the right hand side of (2.26). This leads to

$$\begin{aligned} e^{iW[J]} &= N \int d\Phi d\bar{C} dC e^{i(S_{\text{eff}}[\Phi, \bar{C}, C; \chi[\Phi] + \delta_\epsilon \chi[\Phi]] + J_i \Phi^i)} \\ &\times \left( 1 + i J_i \Omega^i[\Phi] \right) , \end{aligned} \quad (2.32)$$

where  $\delta_\epsilon \chi^\alpha[\Phi] = \epsilon \chi^\alpha_{,i}[\Phi] \Omega^i[\Phi]$ . The change of the gauge conditions in  $S_{\text{eff}}$  can be compensated by a BRST-like transformation (2.29) with  $\lambda = \bar{C}_\alpha \delta_\epsilon \chi^\alpha[\Phi]$ . As a result, one obtains a new realization of the Ward identity (2.20),

$$\Gamma_{,i}[\phi] \langle \Omega^i[\Phi] \rangle = - \Gamma_{,i}[\phi] \langle R^i{}_\alpha[\Phi] C^\alpha \bar{C}_\beta \chi^\beta{}_{,j}[\Phi] \Omega^j[\Phi] \rangle . \quad (2.33)$$

Similarly, the BRST counterparts of eqs. (2.23) and (2.25) are obtained via the substitution  $(F^{-1}[\Phi])^\alpha{}_\beta \rightarrow -C^\alpha \bar{C}_\beta$ . In the BRST approach, it is worth pointing out that the transformation (2.22), which leaves  $S_{\text{GF}}[\chi]$  invariant, becomes

$$\delta_\Lambda \chi^\alpha = \epsilon \Lambda^\alpha{}_\beta \chi^\beta , \quad \delta_\Lambda C^\alpha = 0 , \quad \delta_\Lambda \bar{C}_\alpha = -\epsilon \bar{C}_\beta \Lambda^\beta{}_\alpha , \quad (2.34)$$

which does not change the functional  $S_{\text{GF}}[\chi[\Phi]] + \bar{C}_\alpha \chi^\alpha_{,i}[\Phi] R^i{}_\beta[\Phi] C^\beta$ .

### 3 Conformal Ward identity in $\mathcal{N} = 4$ SYM

As is well known, the  $\mathcal{N} = 4$  super Yang-Mills theory can be obtained by plain dimensional reduction from super Yang-Mills theory in ten dimensions:

$$S = -\frac{1}{4g^2} \int d^{10}x \, \text{tr} \left( F^{MN} F_{MN} - 2i \bar{\Psi} \Gamma^M D_M \Psi \right) , \quad (3.1)$$

where  $F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]$ . In the present paper, we are interested in the bosonic sector of the  $\mathcal{N} = 4$  super Yang-Mills theory described by fields  $A_M = (A_m, Y_\mu)$ , where  $m = 0, 1, 2, 3$  and  $\mu = 1, \dots, 6$ . The classical action reads

$$S[A, Y] = -\frac{1}{4g^2} \int d^4x \, \text{tr} \left( F^{mn} F_{mn} + 2D^m Y_\mu D_m Y_\mu - [Y_\mu, Y_\nu] [Y_\mu, Y_\nu] \right) , \quad (3.2)$$

with  $D_m = \partial_m + iA_m$ , and is invariant under standard gauge transformations

$$\delta A_m = -D_m \tau = -\partial_m \tau - i[A_m, \tau] , \quad \delta Y_\mu = i[\tau, Y_\mu] . \quad (3.3)$$

The action (3.2) is also invariant under arbitrary conformal transformations

$$-\delta_c A_m = \xi A_m + \hat{K}_m{}^n A_n + \sigma A_m , \quad -\delta_c Y_\mu = \xi Y_\mu + \sigma Y_\mu , \quad (3.4)$$

where  $\xi = \xi^m \partial_m$  is a conformal Killing vector field,

$$\partial_m \xi_n + \partial_n \xi_m = 2\eta_{mn} \sigma , \quad \sigma \equiv \frac{1}{4} \partial_m \xi^m , \quad \hat{K}_{mn} \equiv \frac{1}{2} (\partial_m \xi_n - \partial_n \xi_m) . \quad (3.5)$$

The general solution to the conformal Killing equation is

$$\xi^m = a^m + \lambda x^m + K^m{}_n x^n + b^m x^2 - 2x^m (b \cdot x) , \quad K_{mn} = -K_{nm} , \quad (3.6)$$

where

$$\sigma = \lambda - 2(b \cdot x) . \quad (3.7)$$

Our goal below will be to analyse how these conformal transformations (3.4) are deformed at the quantum level.

We will quantize the  $\mathcal{N} = 4$  SYM theory in the framework of the background field method (see [13, 20, 21, 22, 23, 24] and references therein), by splitting the dynamical variables  $\Phi^i = (A_m, Y_\mu)$  into the sum of *background* fields  $\phi^i = (\mathcal{A}_m, \mathcal{Y}_\mu)$  and *quantum* fields  $\varphi^i = (a_m, y_\mu)$ . The classical action  $S[\phi + \varphi] = S[\mathcal{A} + a, \mathcal{Y} + y]$  is then invariant under *background* gauge transformations

$$\begin{aligned} \delta \mathcal{A}_m &= -\mathcal{D}_m \tau , & \delta Y_\mu &= i[\tau, \mathcal{Y}_\mu] , \\ \delta a_m &= i[\tau, a_m] , & \delta y_\mu &= i[\tau, y_\mu] ; \end{aligned} \quad (3.8)$$

and *quantum* gauge transformations

$$\begin{aligned} \delta \mathcal{A}_m &= 0 , & \delta \mathcal{Y}_\mu &= 0 , \\ \delta a_m &= -\mathcal{D}_m - i[a_m, \tau] , & \delta y_\mu &= i[\tau, \mathcal{Y}_\mu + y_\mu] , \end{aligned} \quad (3.9)$$

with  $\mathcal{D}_m$  the background covariant derivatives. The background field quantization procedure consists of fixing the quantum gauge freedom while keeping the background gauge invariance intact by means of background covariant gauge conditions. The effective action is given by the sum of all 1PI Feynman graphs which are vacuum with respect to the quantum fields.

In 't Hooft gauge, the gauge conditions  $\chi^\alpha$  are

$$\chi = \mathcal{D}^m a_m + i[\mathcal{Y}_\mu, y_\mu] , \quad (3.10)$$

and the gauge fixing functional,  $S_{\text{GF}}$ , is

$$S_{\text{GF}}[\chi] = -\frac{1}{2g^2} \int d^4x \, \text{tr} \chi^2 . \quad (3.11)$$

Under the quantum gauge transformations (3.9),

$$\delta_{\text{quantum}}\chi = -\mathcal{D}^m(\mathcal{D}_m\tau + i[a_m, \tau]) + [\mathcal{Y}_\mu, [\mathcal{Y}_\mu + y_\mu, \tau]] \equiv \Delta\tau . \quad (3.12)$$

Here,  $\Delta$  is the Faddeev-Popov operator, denoted by  $F[\Phi]$  in the previous section. Let us introduce a generating functional  $W[\phi; J]$  by the rule

$$e^{iW[\phi; J]} = \int d\varphi \text{Det}\Delta e^{i(S[\phi+\varphi] + S_{\text{GF}}[\chi[\phi; \varphi]] + J \cdot \varphi)} , \quad (3.13)$$

with  $J = (j_m, k_\mu)$  the sources corresponding to  $\varphi = (a_m, y_\mu)$ , and define

$$\langle \varphi \rangle = \frac{\delta}{\delta J} W[\phi; J] . \quad (3.14)$$

In terms of the Legendre transform of  $W[\phi; J]$ ,

$$\Gamma[\phi; \langle \varphi \rangle] = (W[\phi; J] - J \cdot \langle \varphi \rangle)|_{J=J[\phi; \langle \varphi \rangle]} , \quad (3.15)$$

the effective action is

$$\Gamma[\phi] = \Gamma[\phi; \langle \varphi \rangle = 0] . \quad (3.16)$$

Using the conformal transformation laws (3.4), the gauge condition  $\chi$  defined in (3.10) changes as [6, 5]

$$\delta_c\chi = -\xi\chi - 2\sigma\chi + 2(\partial^m\sigma)a_m \equiv \Lambda\chi + \hat{\delta}_c\chi , \quad \hat{\delta}_c\chi = 2(\partial^m\sigma)a_m \quad (3.17)$$

under a combined conformal transformation of the background and quantum fields, while

$$\delta_c S_{\text{GF}}[\chi] = -\frac{1}{g^2} \int d^4x \text{tr}(\chi \hat{\delta}_c\chi) . \quad (3.18)$$

As can be seen, the inhomogeneous part,  $\hat{\delta}_c\chi$ , of the variation  $\delta_c\chi$  makes  $S_{\text{GF}}[\chi]$  conformally non-invariant. Since  $\partial_m\sigma = -2b_m$ , it is in fact the special conformal transformations which render  $S_{\text{GF}}[\chi]$  non-invariant. From (3.17), one also observes that the Faddeev-Popov determinant changes by the rule

$$\text{Det}\Delta \longrightarrow \text{Det}(\Delta + \hat{\delta}_c\Delta) , \quad \hat{\delta}_c\Delta\tau = -2(\partial^m\sigma)(\mathcal{D}_m\tau + i[a_m, \tau]) . \quad (3.19)$$

As a result, we have precisely the situation studied in the previous section, and can therefore make use of the techniques described there.

We will evaluate the variation  $W[\phi + \delta_c\phi; J] - W[\phi; J]$  induced by a conformal transformation  $\phi \rightarrow \phi + \delta_c\phi$  of the background fields, with  $\delta_c\phi$  as in eq. (3.4), in the case when

$\langle\varphi\rangle = 0$ . Using the path integral representation of  $W[\phi + \delta_c\phi; J]$ , as in eq. (3.13), we change the integration variables by the rule  $\varphi \rightarrow \varphi + \delta_c\varphi$ . This gives

$$e^{iW[\phi + \delta_c\phi; J]} = \int d\varphi \text{Det}(\Delta + \hat{\delta}_c\Delta) e^{i(S[\phi + \varphi] + S_{\text{GF}}[\chi + \hat{\delta}_c\chi] + J \cdot \varphi)} . \quad (3.20)$$

The small deformation of the gauge conditions in this expression can be compensated by a field dependent gauge transformation, according to the rules described in the previous section. This results in the following conformal Ward identity

$$\begin{aligned} & \delta_c \mathcal{A}_m \frac{\delta\Gamma[\phi]}{\delta\mathcal{A}_m} + \delta_c \mathcal{Y}_\mu \frac{\delta\Gamma[\phi]}{\delta\mathcal{Y}_\mu} \\ &= -\langle D_m \frac{1}{\Delta} \hat{\delta}_c\chi \rangle \frac{\delta\Gamma[\phi; \langle\varphi\rangle]}{\delta\langle a_m \rangle} \Big|_{\langle\varphi\rangle=0} + i \langle [\frac{1}{\Delta} \hat{\delta}_c\chi, \mathcal{Y}_\mu + y_\mu] \rangle \frac{\delta\Gamma[\phi; \langle\varphi\rangle]}{\delta\langle y_\mu \rangle} \Big|_{\langle\varphi\rangle=0} , \end{aligned} \quad (3.21)$$

with  $D_m = \mathcal{D}_m + i a_m$ . To complete the analysis, we have to express  $\delta\Gamma[\phi; \langle\varphi\rangle]/\delta\langle\varphi\rangle$  at  $\langle\varphi\rangle = 0$  via  $\delta\Gamma[\phi]/\delta\phi$ .

Given an infinitesimal change  $\delta\phi$  of the background fields, let us evaluate the variation  $W[\phi + \delta\phi; J] - W[\phi; J]$  at  $\langle\varphi\rangle = 0$ . Using the path integral representation of  $W[\phi + \delta\phi; J]$ , as in eq. (3.13), we change the integration variables in the manner  $\varphi \rightarrow \varphi - \delta\phi$ . This gives

$$e^{iW[\phi + \delta\phi; J]} = \int d\varphi \text{Det}(\Delta + \delta\Delta) e^{i(S[\phi + \varphi] + S_{\text{GF}}[\chi + \delta\chi] + J \cdot (\varphi - \delta\phi))} , \quad (3.22)$$

where

$$\delta\chi = -D^m \delta\mathcal{A}_m - i [\mathcal{Y}_\mu + y_\mu, \delta\mathcal{Y}_\mu] \quad (3.23)$$

and  $\delta\Delta$  is the deformation in  $\Delta$  induced by  $\delta\chi$ . The change of the gauge conditions in (3.22) can be compensated by a field dependent gauge transformation, as in the preceding section. Then one gets (a similar relation was derived by Hart [24])

$$\begin{aligned} & \delta\mathcal{A}_m \frac{\delta\Gamma[\phi]}{\delta\mathcal{A}_m} + \delta\mathcal{Y}_\mu \frac{\delta\Gamma[\phi]}{\delta\mathcal{Y}_\mu} \\ &= \left\{ \delta\mathcal{A}_m + \langle D_m \frac{1}{\Delta} (D^n \delta\mathcal{A}_n + i [\mathcal{Y}_\nu + y_\nu, \delta\mathcal{Y}_\nu]) \rangle \right\} \frac{\delta\Gamma[\phi; \langle\varphi\rangle]}{\delta\langle a_m \rangle} \Big|_{\langle\varphi\rangle=0} \\ &+ \left\{ \delta\mathcal{Y}_\mu + i \langle [\mathcal{Y}_\mu + y_\mu, \frac{1}{\Delta} (D^n \delta\mathcal{A}_n + i [\mathcal{Y}_\nu + y_\nu, \delta\mathcal{Y}_\nu]) \rangle \right\} \frac{\delta\Gamma[\phi; \langle\varphi\rangle]}{\delta\langle y_\mu \rangle} \Big|_{\langle\varphi\rangle=0} . \end{aligned} \quad (3.24)$$

This relation allows  $\delta\Gamma[\phi; \langle\varphi\rangle]/\delta\langle\varphi\rangle$  at  $\langle\varphi\rangle = 0$  to be expressed in terms of  $\delta\Gamma[\phi]/\delta\phi$  and should be used in conjunction with the Ward identity (3.21). It should be pointed out that our conformal Ward identity given by eqs. (3.21) and (3.24) is more general than that derived in [5].

The conformal Ward identity can be obtained in the BRST approach following the rules given in the previous section. We do not pursue this approach here, but for completeness give the expression for  $S_{\text{eff}}$  and the corresponding BRST transformation. The action  $S_{\text{eff}}$  reads

$$\begin{aligned} S_{\text{eff}} &= S[\phi + \varphi] + S_{\text{GF}}[\chi[\phi; \varphi]] + S_{\text{GH}}[\phi; \varphi, \bar{C}, C] , \\ S_{\text{GF}} &= -\frac{1}{2g^2} \int d^4x \text{tr} \chi^2 , \quad S_{\text{GH}} = \int d^4x \text{tr} \bar{C} \Delta C , \end{aligned} \quad (3.25)$$

and is invariant under the *quantum* BRST transformation

$$\begin{aligned} \delta a_m &= -(\mathcal{D}_m C + i[a_m, C]) \lambda , & \delta y_\mu &= i[C, \mathcal{Y}_\mu + y_\mu] \lambda , \\ \delta C &= -i C^2 \lambda , & \delta \bar{C} &= -\frac{1}{g^2} \chi \lambda . \end{aligned} \quad (3.26)$$

If the ghost fields possess the conformal transformation rules [6]

$$-\delta_c \bar{C} = \xi \bar{C} + 2\sigma \bar{C} , \quad -\delta_c C = \xi C , \quad (3.27)$$

the same conclusion follows from the analysis of the preceding section.

## 4 One-loop calculations

In this section, we compute the leading quantum deformation of the conformal transformation laws of the fields  $\mathcal{A}_m$  and  $\mathcal{Y}_\mu$ . Since there is an overall factor of  $1/g^2$  multiplying the action  $S + S_{\text{GF}}$ , the loop-counting parameter is  $g^2$ . We will examine the conformal Ward identity at the one loop level or, equivalently, to order  $g^2$ .

For the purpose of loop calculations, we expand the action  $S[\phi + \varphi]$  in powers of the quantum fields  $\varphi$  and combine its quadratic part,  $S_2$ , with the gauge fixing functional,  $S_{\text{GF}}$ . This gives

$$\begin{aligned} S_2 + S_{\text{GF}} &= -\frac{1}{g^2} \int d^4x \text{tr} \left\{ \frac{1}{2} a^m \tilde{\Delta} a_m + i \mathcal{F}^{mn} [a_m, a_n] + 2i (\mathcal{D}^m \mathcal{Y}_\mu) [a_m, y_\mu] \right. \\ &\quad \left. + \frac{1}{2} \mathcal{Y}_\mu \tilde{\Delta} \mathcal{Y}_\mu - [\mathcal{Y}_\mu, \mathcal{Y}_\nu] [y_\mu, y_\nu] \right\} , \end{aligned} \quad (4.1)$$

where  $\mathcal{F}_{mn}$  is the background field strength,  $[\mathcal{D}_m, \mathcal{D}_n] = i \mathcal{F}_{mn}$ , and the operator  $\tilde{\Delta}$ ,

$$\tilde{\Delta} \tau = -\mathcal{D}^m \mathcal{D}_m \tau + [\mathcal{Y}_\mu, [\mathcal{Y}_\mu, \tau]] , \quad (4.2)$$

is simply the Faddeev-Popov operator  $\Delta$  in eq. (3.12), evaluated at  $a_m = y_\mu = 0$ . The last term in the right hand side of (4.1) vanishes for an Abelian background.

In the action (4.1), the trace is in the fundamental representation with the generators  $T_i^F$  normalized so that  $\text{tr}(T_i^F T_j^F) = \delta_{ij}$ . Later, we will have need to use the adjoint representation  $(T_i)_j^k = -i f_{ij}^k$ , where the structure constants are defined by  $[T_i^F, T_j^F] = i f_{ij}^k T_k^F$ . The adjoint representation matrices then satisfy the normalization condition  $\text{tr}(T_i T_j) = 2N \delta_{ij}$  for gauge group  $SU(N)$ .

As mentioned earlier, the classical  $\mathcal{N} = 4$  super Yang-Mills action is derived by plain dimensional reduction from the ten-dimensional super Yang-Mills action (3.1). To simplify quantum calculations in the  $\mathcal{N} = 4$  super Yang-Mills theory, it is convenient to restore ten-dimensional notation, as this allows a unified treatment of  $\mathcal{A}_m$  and  $\mathcal{Y}_\mu$ . In ten-dimensional notation, the background fields are  $\mathcal{A}_M = (\mathcal{A}_m, \mathcal{Y}_\mu)$  and the quantum fields are  $a_M = (a_m, y_\mu)$ . The full covariant derivatives  $D_M$  are defined by  $D_M \phi = \partial_M \phi + i [\mathcal{A}_M + a_M, \phi]$ , with  $\partial_M = (\partial_m, 0)$ , and the background covariant derivatives  $\mathcal{D}_M \phi = \partial_M \phi + i [\mathcal{A}_M, \phi]$  define the background field strength  $\mathcal{F}_{MN}$  via  $[\mathcal{D}_M, \mathcal{D}_N] = i \mathcal{F}_{MN}$ . The components of  $\mathcal{F}_{MN}$  are

$$\mathcal{F}_{mn} = \partial_m \mathcal{A}_n - \partial_n \mathcal{A}_m + i [\mathcal{A}_m, \mathcal{A}_n] , \quad \mathcal{F}_{m\nu} = \mathcal{D}_m \mathcal{Y}_\nu , \quad \mathcal{F}_{\mu\nu} = i [\mathcal{Y}_\mu, \mathcal{Y}_\nu] .$$

The gauge-fixing condition (3.10) can be expressed  $\chi = \mathcal{D}^M a_M$ . The action (4.1) becomes (with the ten-dimensional metric containing most pluses)

$$\begin{aligned} S_2 + S_{\text{GF}} &= -\frac{1}{2g^2} \int d^4x \text{tr} \left\{ a^M \tilde{\Delta} a_M - 2i a_M [\mathcal{F}^{MN}, a_N] \right\} , \\ &= -\frac{1}{2g^2} \int d^4x a^{Mi} \left( \tilde{\Delta} \delta_M^N - 2i \mathcal{F}_M^N \right)_i^j a_{Nj} , \end{aligned} \quad (4.3)$$

where the operator  $\tilde{\Delta}$  is just the covariant d'Alembertian

$$\tilde{\Delta} = -\mathcal{D}^M \mathcal{D}_M . \quad (4.4)$$

In the second line of eq. (4.3) and below, we adopt a notation in which background fields are matrices in the adjoint acting on quantum fields which are adjoint vectors. From eq. (4.3), we can read off the propagator

$$\langle a_{Mi}(x) a^{Nj}(x') \rangle = -i \left( \frac{g^2}{\tilde{\Delta} - 2i \mathcal{F}} \right)_{Mi}^{Nj} \delta^4(x, x') . \quad (4.5)$$

Now, we are prepared to analyse quantum conformal invariance. Using (3.17), the conformal Ward identity (3.21) rewritten in ten-dimensional notation takes the form

$$0 = \delta_c \mathcal{A}_{Mi} \frac{\delta \Gamma[\phi]}{\delta \mathcal{A}_{Mi}} + 2 (\partial^n \sigma) \langle (D_M \Delta^{-1})_i^j a_{nj} \rangle \frac{\delta \Gamma[\phi; \langle \varphi \rangle]}{\delta \langle a_{Mi} \rangle} \Big|_{\langle \varphi \rangle=0} . \quad (4.6)$$

As can be seen, the deformation of the conformal transformation law is determined by the average  $\langle (D_M \Delta^{-1})_i^j a_{nj} \rangle$ , which we will evaluate at the one-loop level, i.e. to order  $g^2$ . In this case, the quantum field  $a_{nj}$  must be contracted either with a quantum field in  $D_M$  or with a quantum field in  $\Delta^{-1}$ , with the remaining (uncontracted) quantum fields to be set to zero. The resulting conformal Ward identity is

$$0 = 2i(\partial^n \sigma)(\delta_M^Q + \mathcal{D}_M \tilde{\Delta}^{-1} \mathcal{D}^Q) \langle a_Q^k (T_k \tilde{\Delta}^{-1})_i^j a_{nj} \rangle \frac{\delta\Gamma[\phi; \langle \varphi \rangle]}{\delta \langle a_{Mi} \rangle} \Big|_{\langle \varphi \rangle=0} + \delta_c \mathcal{A}_{Mi} \frac{\delta\Gamma[\phi]}{\delta \mathcal{A}_{Mi}} + O(g^4), \quad (4.7)$$

with the group generators  $T_k$  in the adjoint representation.

The result in equation (3.24) can now be used to express the piece of (4.7) containing  $\delta\Gamma[\phi; \langle \varphi \rangle]/\delta \langle a_{Mi} \rangle$  at  $\langle \varphi \rangle = 0$  in terms of  $\delta\Gamma[\phi]/\delta \mathcal{A}_{Mi}$ . Equation (3.24) states

$$(\delta_M^Q + \langle D_M \Delta^{-1} D^Q \rangle) \delta \mathcal{A}_{Qi} \frac{\delta\Gamma[\phi; \langle \varphi \rangle]}{\delta \langle a_{Mi} \rangle} \Big|_{\langle \varphi \rangle=0} = \delta \mathcal{A}_{Mi} \frac{\delta\Gamma[\phi]}{\delta \mathcal{A}_{Mi}}. \quad (4.8)$$

Since we are working only to order  $g^2$ , if the order  $g^2$  expression

$$\delta \mathcal{A}_{Qi} = 2i(\partial^n \sigma) \langle a_Q^k (T_k \tilde{\Delta}^{-1})_i^j a_{nj} \rangle$$

is substituted into (4.8), then the operator  $(\delta_M^Q + \langle D_M \Delta^{-1} D^Q \rangle)$  acting on  $\delta \mathcal{A}_Q$  is only required at the tree level (order  $g^0$ ), in which case it becomes

$$\delta_M^Q + \mathcal{D}_M \tilde{\Delta}^{-1} \mathcal{D}^Q.$$

This is precisely the background-covariant transverse *projection operator* which appears in the order  $g^2$  term in the conformal Ward identity (4.7). Thus we can use the result (4.8) to rewrite the conformal Ward identity (4.7) in the form

$$0 = \left\{ \delta_c \mathcal{A}_{Mi} + 2i(\partial^n \sigma) \langle a_M^k (T_k \tilde{\Delta}^{-1})_i^j a_{nj} \rangle \right\} \frac{\delta\Gamma[\phi]}{\delta \mathcal{A}_{Mi}} + O(g^4). \quad (4.9)$$

It remains to compute

$$\langle a_M^k (T_k \tilde{\Delta}^{-1})_i^j a_{nj} \rangle.$$

Since in the adjoint representation  $(T_k)_i^j = -i f_{ki}^j = -(T_i)_k^j$ , this can be expressed as

$$-(T_i \tilde{\Delta}^{-1})_k^j \langle a_{nj} a_M^k \rangle,$$

where there is an implicit functional trace. Expanding the propagator (4.5) in powers of the background field strength,

$$\langle a_M^k (T_k \tilde{\Delta}^{-1})_i^j a_{nj} \rangle = ig^2 \eta_{Mn} \text{tr} (T_i \tilde{\Delta}^{-2}) - 2g^2 \text{tr} (T_i \tilde{\Delta}^{-2} \mathcal{F}_{nM} \tilde{\Delta}^{-1}) + O(\mathcal{F}^2). \quad (4.10)$$

Up to this point, the background fields have been completely arbitrary. From now on, we take the gauge group to be  $SU(N+1)$  and restrict attention to a specific Abelian background which is of interest in the context of the AdS/CFT duality. For a single D3-brane probe separated from a stack of  $N$  D3-branes, the relevant background is

$$\mathcal{A}_m = A_m T_0, \quad \mathcal{Y}_\mu = Y_\mu T_0, \quad (4.11)$$

where  $A_m$  is the  $U(1)$  gauge field on the world-volume of the probe brane, and the world-volume scalars  $Y_\mu$  are the components of the transverse separation of the probe brane from the stack of branes. The adjoint generator  $T_0$  corresponds to the following  $su(N+1)$  generator

$$T_0^F = \frac{1}{\sqrt{N(N+1)}} \text{diag}(-1, \dots, -1, N),$$

and its explicit form is

$$T_0 = \sqrt{\frac{N+1}{N}} \text{diag}(0, \dots, 0, 1, -1, \dots, 1, -1),$$

where there are  $N^2$  zeroes on the diagonal and  $N$  pairs  $(1, -1)$ .

For the background chosen, the first term on the right hand side of (4.10) can be neglected; as will be shown later, it leads to a pure gauge transformation. Let us therefore concentrate on the second term. Restricting attention to modifications to the conformal transformation of  $\mathcal{A}_M$  which contain at most one derivative (and hence the  $O(\mathcal{F}^2)$  corrections in (4.10) can be neglected), the functional trace is expressed in momentum space as

$$\int \frac{d^4 k}{(2\pi)^4} \text{tr} \left( T_i \frac{1}{(k^2 + \mathcal{Y}_\nu \mathcal{Y}_\nu)^2} \mathcal{F}_{nM} \frac{1}{(k^2 + \mathcal{Y}_\nu \mathcal{Y}_\nu)} \right).$$

The result in this case is that, using (4.9), the leading quantum deformation of the conformal transformation properties of  $\mathcal{A}_{Mi}$  is

$$\hat{\delta}_c \mathcal{A}_{Mi} = \frac{g^2}{8\pi^2} (\partial^n \sigma) \frac{F_{nM}}{Y^2} \frac{N}{N+1} \text{tr}(T_i T_0).$$

The trace is in the adjoint representation, and ensures that only the nonvanishing components of the background receive a quantum modification:

$$\hat{\delta}_c A_M = \frac{Ng^2}{4\pi^2} (\partial^n \sigma) \frac{F_{nM}}{Y^2}. \quad (4.12)$$

Reducing to four-dimensional notation, this yields

$$\hat{\delta}_c A_m = -\frac{Ng^2}{4\pi^2} (\partial^n \sigma) \frac{F_{mn}}{Y^2}, \quad \hat{\delta}_c Y_\mu = \frac{Ng^2}{4\pi^2} (\partial^n \sigma) \frac{\partial_n Y_\mu}{Y^2}. \quad (4.13)$$



The deformation  $\hat{\delta}_c Y_\mu$  was computed previously in [5].

Let us finally return to the first term on the right hand side of (4.10). It results in a deformation to the conformal transformation of  $\mathcal{A}_{mi}$  of the form  $-2g^2(\partial_m\sigma)\text{tr}(T_i\tilde{\Delta}^{-2})$ . If only the terms which are at most linear in derivatives of the background fields are retained, this can be expressed in momentum space as

$$\begin{aligned} (\partial_m\sigma) \int \frac{d^4k}{(2\pi)^4} \text{tr} \left( T_i \frac{1}{(k^2 + \mathcal{Y}_\nu \mathcal{Y}_\nu)^2} \right) &= \partial_m \left\{ \sigma \int \frac{d^4k}{(2\pi)^4} \text{tr} \left( T_i \frac{1}{(k^2 + \mathcal{Y}_\nu \mathcal{Y}_\nu)^2} \right) \right\} \\ &+ 4\sigma \int \frac{d^4k}{(2\pi)^4} \text{tr} \left( T_i \mathcal{Y}_\nu \partial_m \mathcal{Y}_\nu \frac{1}{(k^2 + \mathcal{Y}_\nu \mathcal{Y}_\nu)^3} \right). \end{aligned}$$

The momentum integral in the first term on the right hand side is ultraviolet divergent; however, the overall derivative means that the contribution is pure gauge and so can be ignored. The second term vanishes on group theoretic grounds, because the adjoint representation is non-chiral,  $\text{tr}(T_i \{T_j, T_k\}) = 0$ .

## 5 Discussion

In order to put the result (4.13) in the context of AdS/CFT correspondence, consider the bosonic action of a single D3-brane probe moving near the core of the stack of  $N$  D3-branes (we set  $2\pi\alpha' = 1$  and ignore the Chern-Simons term, see, e.g. [25] for more detail):

$$S = -T_3 \int d^4x \left( \sqrt{-\det \left( \frac{Y^2}{R^2} \eta_{mn} + \frac{R^2}{Y^2} \partial_m Y_\mu \partial_n Y_\mu + F_{mn} \right)} - \frac{Y^4}{R^4} \right), \quad (5.1)$$

where  $Y^2 = Y_\mu Y_\mu$ ,  $T_3 = 1/g^2$  and  $R^4 = N g^2/(2\pi^2)$ . The action is invariant under the  $AdS_5 \times S^5$  field transformations [1, 26]

$$\delta A_m = \delta_c A_m - \frac{R^4}{2Y^2} (\partial^n \sigma) F_{mn} + \partial_m \left( \frac{R^4}{2Y^2} (\partial^n \sigma) A_n \right), \quad (5.2)$$

$$\delta Y_\mu = \delta_c Y_\mu + \frac{R^4}{2Y^2} (\partial^n \sigma) \partial_n Y_\mu, \quad (5.3)$$

with  $\delta_c A_m$  and  $\delta_c Y_\mu$  the linear conformal transformations (3.4). The nonlinear terms in (5.2) and (5.3) coincide with the quantum deformation (4.13) except for the total derivative in (5.2). Of course, the latter term is not essential since it generates a pure gauge transformation. However, only if it is retained do the variations  $\delta A_m$  and  $\delta Y_\mu$  provide a representation of the conformal algebra.

Eq. (4.13) constitutes the leading quantum deformation, in the framework of the loop expansion and the derivative expansion, to the conformal transformations of  $A$  and  $Y$ . It would be interesting to analyse higher loop and higher derivative deformations. Of course, the fermions and the ghosts have to be taken into account at higher loops. Independently of what happens at higher loops, the one-loop deformation (4.13) is a remarkable result. In conjunction with the requirement of  $SO(6)$   $R$ -symmetry, the conformal transformation (5.3) is known to uniquely fix the action (5.1) for  $F_{mn} = 0$  [1]. On the super Yang-Mills side, this action results from summing up quantum corrections to all loop orders. We believe that the deformed conformal invariance in the  $\mathcal{N} = 4$  super Yang-Mills theory should be crucial, along with the requirement of nonlinear self-duality [27, 28], for a better understanding of numerous non-renormalization theorems which are predicted by the AdS/CFT conjecture and relate to the explicit structure of the low energy effective action in  $\mathcal{N} = 4$  super Yang-Mills theory (see [29, 30, 31] for a more detailed discussion and additional references).

Along with quantum loop calculations, there is a purely field theoretic problem to classify possible *local* field-dependent deformations of the classical conformal transformation, given by  $\delta_c A$  and  $\delta_c Y$ , such that they provide a nonlinear realization of the conformal algebra. This is an interesting and challenging problem which may be addressed within the local BRST cohomological approach (see [32] for a review). It seems that the AdS deformation defined by eqs. (5.2) and (5.3) is the only nontrivial solution to first order in derivatives. Of course, a similar problem may be formulated in terms of superfields. In the case of  $\mathcal{N} = 1$  supersymmetry, for example, one can start with an Abelian gauge superfield  $V$  and three chiral superfields  $\Phi^i$  and then try to deform their linear superconformal transformations (chosen to leave the free actions invariant) so as to end up with an analogue of eqs. (5.2) and (5.3). For a single chiral scalar superfield  $\Phi$ , this problem has been solved implicitly in [33].

The  $\mathcal{N} = 4$  super Yang-Mills theory can be formulated in  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  superspaces. It is therefore natural to wonder whether there may be efficient superfield extensions of the approach advocated in the present paper. The  $\mathcal{N} = 1$  superfield formulation, although most familiar, does not seem to be useful. The point is that the Yang-Mills gauge transformations are known to be highly nonlinear in  $\mathcal{N} = 1$  superspace. In addition, there is no simple  $\mathcal{N} = 1$  superfield generalization of 't Hooft's gauge, without which one inevitably runs into ugly infrared problems. More promising is the  $\mathcal{N} = 2$  harmonic superspace formulation (see [34] for a review), which is similar, in several respects, to the ordinary component formulation. In  $\mathcal{N} = 2$  harmonic superspace, one has a well elabo-

rated background field method [35] and quite powerful heat kernel techniques [36]. The price to pay here, however, might be the need to be extremely careful when evaluating the relevant supergraphs in order to avoid the appearance of so-called harmonic singularities.

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